A GENERALISATION OF EBERLEIN'S INTEGRAL OVER FUNCTION SPACE(1)

BY
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1. **Introduction.** Let

$$\mathfrak{S} = \{(t_1, t_2, \cdots, t_k) = t: -1 \leqq t_j \leqq 1, 1 \leqq j \leqq k\}$$

be a cube in the k-dimensional Euclidean space. Real multiple power series

(1.1) $$x(t) = \sum x_{n_1 \dots n_k}(t_1)^{n_1} \cdots (t_k)^{n_k}$$

where $n_1, n_2, \cdots, n_k$ assume all non-negative integral values, and the coefficients satisfy the condition

(1.2) $$\| x \|_1 = \sum | x_{n_1 \dots n_k} | < \infty,$$

converge absolutely and uniformly for $t \in \mathfrak{C}$. The set of functions defined by (1.1) and (1.2) may be identified with l_1, regarded as a space of k-fold sequences. That the unit sphere

$$S_\infty = \{x \in l_1 : \| x \|_1 \leqq 1\}$$

is compact in the weak* topology of l_1 is well known (cf. [1, p. 424]). An elementary integral E can be defined for the weak* continuous real functions on S_∞. It can then be extended by the Daniell process; and the extended integral induces a countably additive measure on S_∞. Professor W. F. Eberlein defined E for the case $k = 1$ in [2]. We generalise his results to the case $k \geqq 1$. The extended integral and the corresponding measure on S_∞ will be called the *Eberlein integral* and the *Eberlein measure* respectively.

Application of this integral to a discussion of mechanical quadratures will be made in a later paper.

2. **Notation and preliminaries.** We introduce some notation here. More will be explained when the need for it arises.

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Let

(2.1) $$x = \{x_{n_1 \dots n_k}\}$$

be the sequence of coefficients in (1.1). Setting

(2.2) $$X_n = \{x_{n \dots n_k} : n_1 + \cdots + n_k = n\}$$

we can write (2.1) as a simple sequence

(2.3) $$x = (X_0, X_1, \cdots, X_n, \cdots),$$

the elements of X_n being ordered, say lexicographically. We denote the number of elements in X_n by c_n, so that

(2.4) $$c_n = (k + n - 1)!/((k-1)!n!).$$

When the order is relevant, the members of X_n will be enumerated as $\xi_{n1}, \xi_{n2}, \cdots, \xi_{nc}$. If y is a K-fold sequence ordered similarly to x, the symbols $X_n \pm Y_n$ have unambiguous meanings. If the members of Y are $\eta_{n1}, \eta_{n2}, \cdots, \eta_{nc_n}$ in that order, we write $\langle X_n, Y_n \rangle$ for $\sum_{i=1}^{c_n} \xi_{ni}\eta_{ni}$ and $\langle x, y \rangle$ for $\sum_{n=0}^{\infty} \langle X_n, Y_n \rangle$. The following notation will be used to condense long expressions:

$$\sigma_n = \sum_{i=0}^{n} c_n; \mu_n = \prod_{i=1}^{n} (c_i!); X_n t^n = \sum x_{n_1 \dots n_k} (t_1)^{n_1} \cdots (t_k)^{n_k},$$

the summation being over $n_1, \cdots, n_k$ such that $n_1 + \cdots + n_k = n$;

$$|X_n| = \sum_{i=1}^{c_n} |\xi_{ni}|;$$

$$dX_n = d\xi_{n1} d\xi_{n2} \cdots d\xi_{nc_n};$$

$$\alpha_n = \sum_{i=0}^{n} |X_i|;$$

and

$$\phi_n = \prod_{i=1}^{n} (1 - \alpha_{i-1})^{c_i}.$$

We set down for easy reference, Liouville's generalisation of a theorem of Dirichlet's on multiple integrals [3, p. 160]:

$$\int \cdots \int (u_1)^{a_1} \cdots (u_n)^{a_n} f(u_1 + \cdots + u_n) du_1 \cdots du_n$$

$$= \frac{\Gamma(a_1 + 1) \cdots \Gamma(a_n + 1)}{\Gamma(a_1 + \cdots + a_n + n)} \int_0^c h^{a+n-1} f(h) dh$$

where $a = a_1 + \cdots + a_n$, and the multiple integral is over all the non-negative values of the variables u_i such that their sum is not greater than a prescribed constant c. We shall refer to this result as Dirichlet's theorem.

3. **The Eberlein integral.** We begin with the observation that the unit sphere S_∞ of l_1 is a compact metric space in the weak* topology of l_1. The convergence is pointwise, and a metric is

$$\rho(x,y) = \sum_{n=0}^{\infty} 2^{-n}|X_n - Y_n|/(1+|X_n - Y_n|) \quad (x, y \in S_\infty).$$

The set weak* continuous real functions on S_∞, which we denote by $\mathscr{C}(S_\infty)$, is a Banach algebra under pointwise algebraic operations and the norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in S_\infty\} \quad (f \in \mathscr{C}(S_\infty));$$

and closed under the lattice operations, "max" and "min". If we define an elementary integral for functions of $\mathscr{C}(S_\infty)$, it can be extended in the standard fashion [4, Chapter III] to obtain the Eberlein integral, which in turn induces the countably additive Eberlein measure on S_∞. We address ourselves now to the task of contructing this elementary integral, E.

For $n = 0,1,2,\cdots$, let P_n be the projection operator on l_1:

$$P_n x = (X_0, X_1, \cdots X_n, \mathbf{0}, \mathbf{0}, \cdots)$$

where

$$x = (X_0, X_1, \cdots, X_n, X_{n+1}, \cdots) \in l_1$$

and the $\mathbf{0}$'s in the definition of $P_n x$ denote sets of zeros. A function f on l_1 is a cylinder function of degree n iff $f(x) = f(P_n x)$. Denoting the set of cylinder functions of degree n by L_n, we see that

$$L_0 \subset L_1 \subset \cdots \subset L_n \subset L_{n+1} \subset \cdots \subset L_\infty = \bigcup_n L_n. \tag{3.1}$$

We define the integral E on $\mathscr{C}(S_\infty)$ in such a manner that, for $f \in L_0 \cap \mathscr{C}(S_\infty)$, $E(f)$ reduces to the ordinary average

$$E(f) = \frac{1}{2}\int_{-1}^{1} f\, dX_0 = \frac{1}{2}\int_{-1}^{1} f\, d\xi_{01}. \tag{3.2}$$

Generally, for $f \in L_n \cap \mathscr{C}(S_\infty)$, we define

$$E(f) = \frac{\mu_n}{2^{\sigma_n}} \int\cdots\int_{\alpha_n \leqq 1} \frac{f}{\phi_n}\, dX_0 \cdots dX_n. \tag{3.3}$$

If we use (3.3) for $f \in L_{n-1} \cap \mathscr{C}(S_\infty)$, we have only to use Dirichlet's theorem to perform the integrations with respect to dX_n, to obtain

$$E(f) = \frac{\mu_{n-1}}{2^{\sigma_{n-1}}} \int\cdots\int_{\alpha_{n-1} \leqq 1} \frac{f}{\phi_{n-1}}\, dX_0 \cdots dX_{n-1}, \tag{3.4}$$

which shows that the definition (3.3) is consistent with the inclusions (3.1). If $f \in L_0 \cap \mathscr{C}(S_\infty)$ and we use (3.3), we can perform n reductions like the above and arrive precisely at (3.2). Now we let $f \in \mathscr{C}(S_\infty)$ be arbitrary, and define the functions f_n for $n = 0, 1, 2, \cdots$ by $f_n(x) = f(P_n x)$ for $x \in S_\infty$. Clearly, $\rho(x, P_n x) \leqq 2^{-n} \to 0$ as $n \to \infty$. Since f is continuous on the compact space S_∞, and hence uniformly continuous, we have

$$\|f - f_n\|_\infty = \sup\{|f(x) - f(P_n x)| : x \in S_\infty\} \to 0 \text{ as } n \to \infty.$$

Hence $f_n \to f$ uniformly. As E is obviously a linear functional of unit norm on each subspace $L_n \cap \mathscr{C}(S_\infty)$ of $\mathscr{C}(S_\infty)$,

$$|E(f_m) - E(f_n)| = \|f_m - f_n\|_\infty \to 0 \text{ as } m, n \to \infty.$$

Then $\{E(f_n) : n = 0, 1, 2, \cdots\}$ is a Cauchy sequence of real numbers, and we define $\lim_n E(f_n)$ as $E(f)$. Explicitly, for $f \in \mathscr{C}(S_\infty)$,

$$E(f) = \lim \frac{\mu_n}{2^{\sigma_n}} \int \cdots \int_{\alpha_n \leqq 1} \frac{f(P_n x)}{\phi_n} \, dX_0 \cdots dX_n. \tag{3.5}$$

We remark that this same formula serves to define $E(f)$ for any bounded Baire function f. From here the standard Daniell extension process leads to the Eberlein integral and the Eberlein measure on S_∞. We shall denote the extended integral also by the symbol $\int_{S_\infty} f \, d_E x$.

THEOREM. *The Eberlein measure on S_∞ is concentrated on $\{x \in S_\infty : \|x\|_1 = 1\}$, the strong boundary of S_∞.*

Proof. As the function $(1 - \|x\|_1)$ is positive in the interior and zero on the boundary of S_∞, it is clearly enough to show that

$$\int_{S_\infty} (1 - \|x\|_1) \, d_E x = 0.$$

Noting that $\|x\|_1 = \lim_n \alpha_n$ and that $\alpha_n \in \mathscr{C}(S_\infty)$ for $n = 0, 1, 2, \cdots$, and using (3.5) we have, for $K > -1$,

$$\int_{S_\infty} (1 - \|x\|_1)^K \, d_E x = \lim_n Q_n^K$$

where

$$Q_n^K = (\mu_n / 2^{\sigma_n}) \int \cdots \int_{\alpha_n \leqq 1} [(1 - \alpha_n)^K / \phi_n] \, dX_0 \cdots dX_n.$$

On performing the indicated integrations with respect to dX_n by the aid of Dirichlet's theorem, we see that

$$Q_n^K = \frac{c_n! \, \Gamma(K+1)}{\Gamma(c_n + K + 1)} Q_{n-1} \leqq \frac{1}{K+1} Q_{n-1}^K.$$

Hence

$$0 < Q_n^K \leqq Q_0^K/(K+1)^n = 1/(K+1)^{n+1};$$

and when $K > 0$, $Q_n^K \to 0$ as $n \to \infty$. Q.E.D.

COROLLARY. *When* $K > -1$,

$$Q_n^K = \frac{\mu_n[\Gamma(K+1)]^n}{(K+1)\prod_{i=1}^{n}\Gamma(c_i+K+1)}.$$

4. **Some special integrals.** We evaluate a number of special integrals leading up to (4.19).

(i) We begin with the obvious observation that, when K is an odd positive integer,

$$\int_{S_\infty} (\xi_{ni})^K d_E x = 0. \tag{4.1}$$

(ii) For any real $K > -1$,

$$\int_{S_\infty} |\xi_{ni}|^K d_E x = \frac{\mu_n[\Gamma(K+1)]^n}{(K+1)\prod_{i=1}^{n}\Gamma(c_i+K+1)}. \tag{4.2}$$

By definition, this integral equals

$$\frac{\mu_n}{2^{\sigma_n}} \int \cdots \int_{\alpha \leqq 1} \frac{1}{\phi_n}\{\ \}\, dX_0 \cdots dX_{n-1} \tag{4.3}$$

where

$$\{\ \} = \int_{|X|\leqq 1-\alpha_{n-1}} |\xi_{ni}|^K dX_n.$$

Evaluating this last integral by the use of Dirichlet's theorem and substituting in (4.3), we find the value of the integral in (4.2) to be

$$\frac{c_n!\Gamma(K+1)}{\Gamma(c_n+K+1)}\, Q_{n-1}^K.$$

Substituting for Q_{n-1}^K from (3.6), we obtain the equality (4.2). When K is an integer, (4.2) may be written as

$$\int_{S_\infty} |\xi_{ni}|^K d_E x = \frac{(K!)}{(K+1)\prod_{i=1}^{n}\prod_{l=1}^{K}(c_i+l)}. \tag{4.4}$$

(iii) If n and r are non-negative integers, then

$$\int_{S_\infty} \langle X_n, Y_n\rangle^r d_E x \tag{4.5}$$

equals zero or

$$\frac{(r!)^n}{(r+1)\prod_{i=1}^{n}\prod_{l=1}^{r}(c_i+l)} \Sigma' (\eta_{n1})^{K_1}\cdots(\eta_{nc})^{K_c}, \tag{4.6}$$

according as r is odd or even. Here Σ' denotes that the sum is taken over all non-negative *even* values of the K's such that their sum is equal to r.

Expanding $\langle X_n, Y_n\rangle^r$ by the multinomial theorem and using the linearity of the Eberlein integral, we can write (4.5) as

$$\Sigma' \frac{r!}{K_1!\cdots K_c!}\int_{S_\infty}(\xi_{n1}\eta_{n1})^{K_1}\cdots(\xi_{nc_n}\eta_{nc_n})^{K_{c_n}}d_E x \tag{4.7}$$

where the summation is as explained above. It is obvious that if, in the sum (4.7), any of the K's were odd in a term, the term vanishes. It follows that (4.5) itself vanishes when r is odd. The typical integral appearing in (4.7) equals

$$\frac{\mu_n}{2^{\sigma_n}}\int\cdots\int_{\alpha_n\leqq 1}\frac{1}{\phi_n}\{\ \}dX_0\cdots dX_n \tag{4.8}$$

where

$$\{\ \} = \int_{|X_n|\leqq 1-\alpha\ -:}(\xi_{n1})^{K_1}\cdots(\xi_{nc})^{K_{c_n}}dX_n.$$

Using Dirichlet's theorem to evaluate the last integral and using the evaluation of Q^r_{n-1} from (3.6), we reduce (4.8) to

$$\frac{(r!)^{n-1}K_1!K_2!\cdots K_{c_n}!}{(r+1)\prod_{i=1}^{n}\prod_{l=1}^{r}(c_i+l)}.$$

Using this in the general term of (4.7), we see the equality of (4.5) and (4.6).

(iv) For a fixed non-negative integer n, let $r_0, r_1, \cdots, r_n$ be any non-negative integers. We shall evaluate

$$I = \int_{S_\infty}\langle X_0, Y_0\rangle^{r_0}\cdots\langle X_n, Y_n\rangle^{r_n}d_E x. \tag{4.9}$$

As this integral vanishes whenever one or more of the r's is odd, we assume them all to be even. The following condensed notation will be useful in evaluating the above integral:

$$R_{-1} = 0, \text{ and } R_j = \sum_{i=0}^{j} r_{n-i}(0\leqq j\leqq n);$$

$$N(n,r) = \Sigma'(\eta_{n1})^{K_1}\cdots(\eta_{nc_n})^{K_c},$$

where Σ' means the same as in (4.6); and

$$I_{n-j} = \frac{\mu_{n-j}}{2^{\sigma_{n-j}}} \int_{\alpha_{n-j}\leqq 1} \cdots \int \frac{1}{\phi_{n-j}} \left[\prod_{i=0}^{n-j} \langle X_i, Y_i\rangle^{r_i} \right] (1-\alpha_{n-j})^{R_{j-1}} dX_0 \cdots dX_{n-j}.$$

We rewrite

$$(4.10) \qquad I_{n-j} = \frac{\mu_{n-j}}{2^{\sigma_{n-j}}} \int_{\alpha_{n-j}\leqq 1} \cdots \int \frac{\prod_{i=0}^{n-j-i} \langle X_i, Y_i\rangle^{r_i}}{\phi_{n-j}} \cdot \{ \ \} \, dX_0 \, dX_1 \cdots dX_{n-j-i}$$

where

$$\{ \ \} = \int_{|X_{n-j}| \leqq 1-\alpha_{n-j-1}} \langle X_{n-j}, Y_{n-j}\rangle^{r_{n-j}} . (1-\alpha_{n-j-1} - |X_{n-j}|)^{R_{j-1}} \, dX_{n-j}.$$

Using the multinomial theorem to expand $\langle X_{n-j}, Y_{n-j}\rangle^{r_{n-}}$ and integrating termwise using Dirichlet's theorem, we obtain

$$\{ \ \} = 2^{c_{n-j}} . \frac{r_{n-j}! \, R_{j-1}!}{(R_j + c_{n-j})!} . \, (1-\alpha_{n-j-1})^{R_j + c_{n-j}} . \, N(n-j, r_{n-j}).$$

Substituting this in (4.10), we obtain

$$(4.11) \qquad I_{n-j} = I_{n-j-1} . \frac{r_{n-j}! \, R_{j-1}! \, N(n-j, r_{n-j})}{\prod_{i=1}^{R_j} (c_{n-j} + l)},$$

a reduction formula for I_{n-j}. Repeated application of this formula yields

$$(4.12) \qquad I = I_n = \prod_{i=0}^{n} \frac{r_i! \, R_{n-i-1}! \, N(i, r_i)}{\prod_{l=1}^{R_{n-i}} (c_i + l)}.$$

(v) If y is a bounded k-fold sequence of real numbers, i.e. if $y \in m = (l_1)^*$, then the absolutely convergent series

$$\langle x, y\rangle = \sum_{n=0}^{\infty} \langle X_n, Y_n\rangle$$

defines a bounded Baire function of class 1 on the compact set S_∞ to the real numbers. Hence

$$(4.13) \qquad \int_{S_\infty} \langle x, y\rangle^p \, d_E x$$

exists for every non-negative p. It is clear that the integral reduces to 1 when $p = 0$, and that it vanishes when p is an odd positive integer. First we note that

$$(4.14) \qquad \begin{aligned} \langle x, y\rangle^p &= \left[\sum_{n=0}^{\infty} \langle X_n, Y_n\rangle \right]^p \\ &= \Sigma_1 \, \Sigma_2 \, \Sigma_3 \, \frac{p!}{r_{n_1}! \; r_{n_2}! \cdots r_{n_j}!} \, \langle X_{n_1}, Y_{n_1}\rangle^{r_{n_1}} \cdots \langle X_{n_j}, Y_{n_j}\rangle^{r_{n_j}} \end{aligned}$$

and that term-by-term integration is permissible. Here $\sum_1$ is a sum over j from 1 to p; $\sum_2$ is a sum over all integers $n_1, n_2, \cdots, n_j$ such that $0 \leqq n_1 < n_2 < \cdots n_j < \infty$; and $\sum_3$, a sum over all strictly positive integers $r_{n_1}, r_{n_2}, \cdots, r_{n_j}$ whose sum equals p. As we are going to integrate over S_∞ with respect to $d_E x$, it is obvious that we can also stipulate that each of $r_{n_1}, r_{n_2}, \cdots, r_{n_j}$ be even. To integrate the typical term of (4.14), we can exploit the result (4.12) which gives the value of the integral (4.9). First we have to re-express this term (apart from the constant factor) in the same form as the integrand of (4.9). To this end, we define

$$r_m = 0 \quad (0 \leqq m \leqq n_j\,;\ m \neq n_1, n_2, \cdots, \text{or } n_j) \tag{4.15}$$

and rewrite the summand of (4.14) as

$$\frac{p\,!}{r_0!\, r_1! \cdots r_{n_j}!} \langle X_0, Y_0\rangle^{r_0} \langle X_1, Y_1\rangle^{r_1} \cdots \langle X_{n_j}, Y_{n_j}\rangle^{r_{n_j}} .$$

Using the result of (4.12), we conclude that the integral of the summand in (4.14) is

$$\frac{p!\prod_{i=0}^{n_j} (R_{n_j-i-1}\,!)\cdot \prod_{i=0}^{n_j} N(i, r_i)}{\prod_{i=0}^{n_j} \prod_{l=1}^{R_{n_j-1}} (c_i+l)}. \tag{4.16}$$

This expression, however, involves some extraneous r's introduced through (4.15). Since it is not always expedient to keep track of these vanishing r's, we have to purge the expression (4.16) of its dependence on them. First we note that $N(i, r_i) = 1$ whenever $r_i = 0$, so that we can rewrite the product $\prod_{i=0}^{n_j} N(i, r_i)$ as $\prod_{i=1}^{j} N(n_i, r_{n_i})$. Then we define the quantities

$$P_{j-m} = \sum_{i=m}^{j} (r_{n_i}) \qquad (1 \leqq m \leqq j)$$

so that

$$P_{j-1} = R_{n_j-n_1} = R_{n_j-n_1+1} = \cdots = R_{n_j}$$

and

$$P_i = R_{n_j-n_{j-i}} = R_{n_j-n_{j-i}+1} = \cdots = R_{n_j-n_{j-i-1}-1} \qquad (0 \leqq i \leqq j-2).$$

Using this notation, we can write

$$\prod_{i=0}^{n_j} (R_{n_j-i-1}\,!) = \frac{\prod_{i=1}^{j} (P_{j-i}\,!)^{n_i}}{\prod_{i=2}^{j} (P_{j-i}\,!)^{n_{i-1}}}.$$

In considering

$$\prod_{i=0}^{n_j} \prod_{l=1}^{R_{j}-1} (c_i+l),$$

we take the product over i in different stretches:

$$i=0;\ 1\leqq i\leqq n_1;\ \ n_1+1\leqq i\leqq n_2;\cdots;\ n_{j-1}+1\leqq i\leqq n_j;$$

and obtain

$$\left[\prod_{l=1}^{P_{j-1}}(c_0-l)\right]\left[\prod_{i=1}^{j}\prod_{l=1}^{P_{j-1}}(c_{n_i}+l)(c_{n_i-1}+l)\cdots(c_{n_{i-1}+1}+l)\right].$$

Now we can write (4.16) in the form

(4.17)
$$\frac{p!\prod_{i=1}^{j}(P_{j-i}!)^{n_i}\cdot\prod_{i=1}^{j}(N(n_i,r_{n_i}))}{\left[\prod_{i=2}^{j}=(P_{j-i}!)^{n_i-1}\right]\left[\prod_{l=1}^{P_{j-1}}(c_0+l)\right]\left[\prod_{i=1}^{j}\prod_{l=1}^{P_{j-1}}(c_{n_i}+l)(c_{n_i-1}+l)\cdots(c_{n_{i-1}+1}+l)\right]}.$$

Denoting this expression by the symbol, $\{(4.17)\}$, we get, for the value of the integral (4.13), the expression

(4.18)
$$\sum_1\sum_2\sum_3\{(4.17)\}$$

where Σ_1, Σ_2, Σ_3, mean the same as in (4.14), and we understand that $r_{n_1},r_{n_2},\cdots,r_{n_j}$ are all strictly positive and even.

(vi) Of considerable importance to the discussion of mechanical quadrature formulae is the special case, $p=2$, of the general result (4.18). When $p=2$, the index j in the sum Σ_1 assumes the two values, 1 and 2. When $j=2$, the corresponding sum Σ_3 is empty. Collecting the terms for $j=1$, we get

(4.19)
$$\int_{S_\infty}\langle x,y\rangle^2 d_E x=\frac{1}{3}\sum_{n=0}^{\infty}\frac{2^n\sum_{i=1}^{c}(\eta_{ni})^2}{\prod_{i=1}^{n}(c_i+1)(c_i+2)}.$$

Similarly, when $p=4$,

$$\int_{S_\infty}\langle x,y\rangle^4 d_E x$$

is the sum of

$$\frac{1}{5}\sum_{0\leqq m\leqq n=1}^{\infty}\frac{12^m 2^n\left[\sum_{i=1}^{c}(\eta_{mi})^2\right]\left[\sum_{i=1}^{c}(\eta_{ni})^2\right]}{\left[\prod_{i=1}^{m}\prod_{l=1}^{4}((c_i+l)\right]\left[\prod_{i=m+1}^{n}(c_i+1)(c_i+2)\right]}$$

and

$$\frac{1}{5}\sum_{n=0}^{\infty}\frac{24^n\left[\sum_{i=1}^{c}(\eta_{ni})^2\right]^2}{\prod_{i=1}^{n}\prod_{l=1}^{4}(c_i+l)}.$$

It is clear now that the value of the integral (4.13) can, in principle, be explicitly evaluated for any non-negative integral value of p, though the work can become prohibitively unwieldy for large even values of p.

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